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Quasiperiodic Tilings with Low-Order Rotational Symmetry

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Abstract

Using a projection method of de Bruijn [*Proc. K. Ned. Akad. Wet. Ser. A* (1981), **43**, 39-66], Whittaker & Whittaker [*Acta Cryst.* (1988), A**44**, 105-112] obtained nonperiodic tilings of the plane with n-fold rotational symmetry, $n = 5$, 7, 8, 9, 10 and 12. However, when their method was applied to the cases of 3-, 4- and 6-fold rotational symmetry it produced periodic tilings. This might be taken as circumstantial evidence that 3-, 4- and 6-fold rotational symmetry is incompatible with nonperiodicity. It is demonstrated that they are compatible by constructing quasiperiodic tilings with only 3-, 4- and 6-fold rotational symmetry. This approach uses basic eigenvalue methods of matrix theory.

Introduction

Whittaker & Whittaker (1988) recently used a projection method of de Bruijn (1981) to obtain nonperiodic tilings of the plane with n-fold rotational symmetry, $n = 5$, 7, 8, 9, 10 and 12. Interestingly, when their method was applied to the cases of 3-, 4- and 6-fold rotational symmetry it produced periodic tilings. It might be conjectured that nonperiodicity is incompatible with 3-, 4- and 6-fold rotational symmetry, mirroring the incompatibility of periodicity with 5-fold rotational symmetry. We settle this conjecture here by constructing quasiperiodic tilings with n-fold (but not $2n$-fold) rotational symmetry for $n = 3, 4$ and 6.

Our approach is related to Roger Penrose's inflation method, however, we give a precise definition of inflation and deflation in terms of a matrix eigenvalue problem which determines each tiling. Thus, our secondary motivation is to introduce techniques from linear algebra into the study of quasiperiodic tilings.

We give formal proofs in § 1 for two reasons. Firstly, our methods are elementary, so that the widest possible audience can see what we are doing. Secondly, the relatively new field of nonperiodic tilings is conspicuous in its shortage of proven results (Grünbaum & Shephard, 1987).

The generalized Penrose tilings of Whittaker & Whittaker (1988), as well as the 8-fold rotationally symmetric tilings of Watanabe, Ito & Soma (1987) can all be obtained as special cases using the eigenvalue approach developed in this paper. In fact, the method is quite general.

1. Fourfold rotational symmetry

Recent work in quasicrystals has emphasized 5-fold rotational symmetry for good empirical and theoretical reasons. However, it is also reasonable to investigate quasiperiodic tilings using building blocks more elementary than the relatively exotic Penrose tiles: for example, squares and 45° rhombi.

To construct quasiperiodic tilings of the plane with only 4-fold rotational symmetry we use as prototiles (Grünbaum & Shephard, 1987) unit-edged squares and unit-edged rhombi with 45° vertex angle. These are placed edge to edge in finite assemblies called tiles. Square prototiles are permitted to be cut along a diagonal just so long as the cut-diagonal edge appears on the outer boundary of the tile. This is in anticipation of matching the half-square with a half-square belonging to another tile, thus reproducing a square prototile. Our first tiles (as well as all succeeding generations) will also be squares and rhombi, that is, larger replicas built from the prototiles. The constructions are shown in Fig. 1.

S_1 already has the 4-fold rotational symmetry that will be inherited by all succeeding S_n and, ultimately, by the limiting tiling of the infinite plane. The number of each type of prototile used in R_1 and S_1, as well as the side length $\lambda = 1 + 2^{1/2}$ of both first-generation tiles, are obtained as solutions to the inflation problem, *i.e.* the eigenvalue problem

$$M^2\mathbf{a} = \lambda^2\mathbf{a} \qquad (1)$$

where $\mathbf{a} = (r_1, s_1)^T$ is a given eigenvector of prototile areas $r_1 = \text{area}\,(R_1) = 1/2^{1/2}$, $S_1 = \text{area}\,(S_1) = 1$.

Different from the usual eigenvalue problem, the eigenvector is given in advance and the unknowns are integer elements of the matrix M and the largest eigenvalue λ of M. Solved successfully, (1) becomes a combinatorial area identity specifying how many R_1's and how many S_1's make one R_2 and one S_2

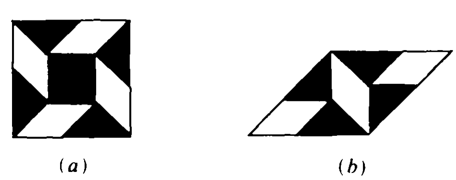

Fig. 1. (*a*) First-generation square S_1. (*b*) First-generation rhombus R_1.

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with side length λ. Note that $r_2 = \lambda^2 r_1$ and $s_2 = \lambda^2 s_1$. The matrix-vector interpretation of Fig. 1 becomes

$$\begin{pmatrix} 3 & 2 \\ 4 & 3 \end{pmatrix} \begin{pmatrix} r_1 \\ s_1 \end{pmatrix} = (1+2^{1/2})^2 \begin{pmatrix} r_1 \\ s_1 \end{pmatrix}, \quad \text{where } M = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}.$$

Fig. 1 is literally a picture of this eigenvalue problem, indicating that three rhombi and two squares build a larger rhombus (with side length $\lambda = 1 + 2^{1/2}$) and that four rhombi and three squares build a larger square. The entries of the matrix M (as opposed to M^2) also have an interesting combinatorial interpretation, brought out in the proof of nonperiodicity, below.

The deflation problem

$$\lambda^{-2} M^2 \mathbf{a} = \mathbf{a} \tag{2}$$

is simply a change in point of view, and a solution to (2) gives numerical information for subdividing a unit square and rhombus into subsquares and subrhombi.

It should be stressed that (1) and (2) are area identities. Self-similar replication of the patterns of Fig. 1 will produce larger and larger squares and rhombi, but we need to prove that these nth-generation objects are valid tiles. The reason is the use of the half-squares on the boundaries. If half-squares fail to match in the interior of the resulting object, it is a violation of the condition that all interior prototiles be either squares or rhombi. For example, Fig. 2 is an illustration of a second-generation rhombus free of such interior violations.

Observe that the edges of the rhombus in Fig. 2 are partitioned into distance units of 1 or $2^{1/2}$, and that every edge sequence of this rhombus is $E_2 = (1, 2^{1/2}, 1, 2^{1/2}, 1)$. In fact, equality of the edge sequences is necessary and sufficient to avoid boundary mismatches throughout the construction process. E_2 has the useful property of palindromy: the sequence is the same when read from right to left as it is when read from left to right. However, this need not be true in general so that the edges of nth-generation tiles must be oriented in order to specify uniquely the edge sequences.

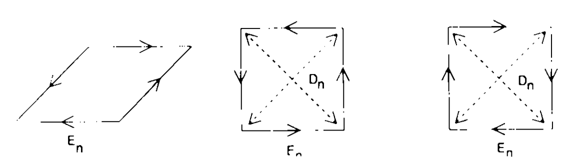
We now give a recursive procedure which generates a quasiperiodic tiling of the plane with 4-fold rotational symmetry. Suppose that at the nth stage of construction we have the nth-generation rhombus and the two versions of the nth-generation square shown in Fig. 3.

Fig. 3(c) is obtained from (b) by reflection about the central vertical axis (and *vice versa*). Let D_n denote the sequence of 1's and $(2^{1/2})$'s read along the diagonals of the nth-generation square of Fig. 3(b). Our working assumption, the induction hypothesis, is that both diagonal sequences are equal and palindromic (signified by the double arrowheads). The reflection that transforms (b) into (c) does not change this. All other edge sequences in Fig. 3, denoted by E_n, are assumed to be equal when read in the direction of the orientations shown in Fig. 3. Note that all the conditions which we are assuming to be true at this stage of construction are true in generation 0 (the prototiles) and generation 1 (Fig. 1). Now we build the next generation's tiles, Fig. 4, using the nth-generation tiles. Half-squares are cut from the square tiles S_n just as they were from the prototiles S_0.

We have all boundary edge sequences $E_{n+1} = (D_n, E_n)$ identical, as well as equal and palindromic diagonal sequences $D_{n+1} = (\overleftarrow{E_n}, D_n, E_n)$, where $\overleftarrow{E_n}$ denotes the edge sequence E_n read in reverse order. We obtain the two versions of the square tile, Figs. 4(b) and (c), with their counterclockwise and clockwise edge orientations, respectively.

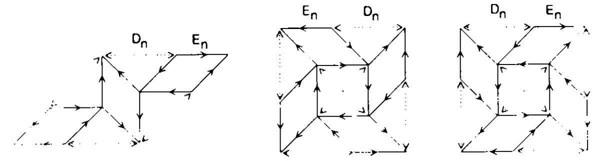
In other words, all the features which were assumed to be present in the nth generation are transmitted successfully to the $(n+1)$th generation. Since these features are present in generations 0 and 1, as remarked above, all generations of tiles have them. That is, we can be certain that there is no unmatched half-square prototile in the interior of any tile in any generation as constructed above. Several generations of growth are shown in Fig. 5.

The tiling of the infinite plane is obtained as the limit of a square tile as $n \to \infty$. This infinite tiling has



(a) (b) (c)

Fig. 3. The induction hypothesis: preparing the transition to the $(n+1)$th generation.



(a) (b) (c)

Fig. 4. The induction step: verifying the induction hypothesis in the $(n+1)$th generation.

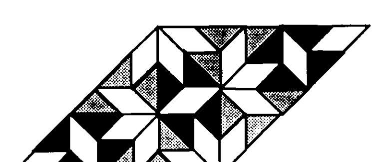


Fig. 2. The second-generation rhombus is a self-similar replica of Fig. 1(b).

4-fold rotational symmetry but, as we show below, it does not have the translational symmetry needed for periodicity. Although not periodic in a translational sense, it does have repetitiveness in a dynamic sense. Each generation has the same large-scale structure shown in Figs. 1 and 4.

To see that the tiling is nonperiodic, suppose, for a contradiction, that it is periodic. Because of the 4-fold rotational symmetry, the tiling must be an infinite 'chessboard' made from sufficiently large (identically patterned) squares. Thus the diagonal sequence D (of terms 1 and $2^{1/2}$, obtained as the limit of D_n) must be a periodic sequence. To show that this is impossible, we use the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix}$ to answer two basic questions: (1) How many terms are there in a diagonal sequence D_n? (2) How many of these terms are 1's? Recall the relations $E_{n+1} = (D_n, E_n)$ and $D_{n+1} = (\overleftarrow{E_n}, D_n, E_n)$. If we let e_n (d_n) stand for the number of terms in E_n (D_n), then clearly

$$\begin{pmatrix} e_{n+1} \\ d_{n+1} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} e_n \\ d_n \end{pmatrix}. \qquad (3)$$

This simple recurrence system is easily solved to get

$$d_n \simeq (1/2 + 1/2^{1/2})(1+2^{1/2})^n,$$

where $\simeq$ denotes equality when rounded off to the nearest integer. For example, when $n=4$ we have $d_4 = 41$ and one counts exactly 41 prototiles along the diagonal in Fig. 5. Similarly, if we let ε_n and δ_n denote the number of 1's in E_n and D_n, respectively, then ε_n and δ_n also satisfy (3), but with different initial values $\varepsilon_1 = 1$ and $\delta_1 = 2$ (as opposed to $e_1 = 2$ and $d_1 = 3$).

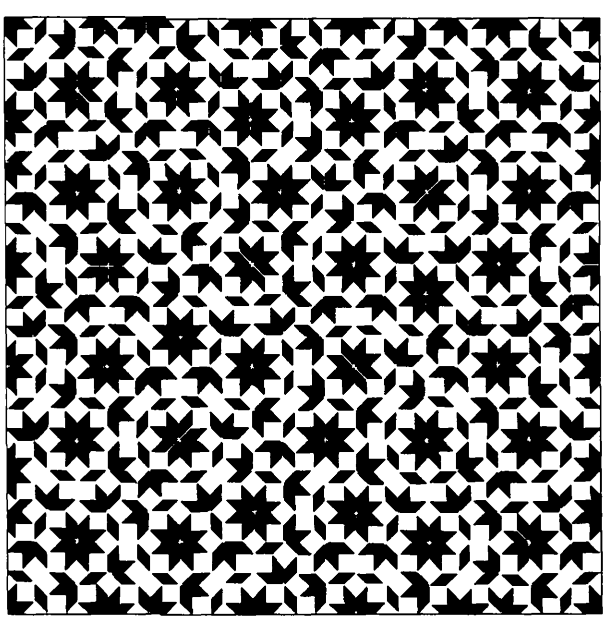


Fig. 5. Fourth-generation square with 4-fold rotational symmetry.

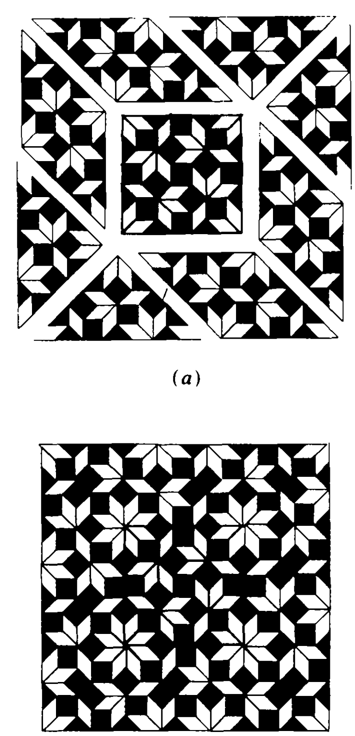
For $n \geq 1$,

$$\delta_n \simeq (1/2^{1/2})(1+2^{1/2})^n.$$

If the diagonal sequence D is periodic, the ratio δ_n/d_n should approach a limit which is a rational number as n tends to infinity. However, this ratio approaches the irrational number $2 - 2^{1/2}$, proving that D cannot be periodic. A more familiar argument would be to show that the ratio of squares to rhombi tends in the limit to an irrational number; but here we wanted to bring out the role of the matrix M.

It is also easy to see why there are uncountably infinitely many different tilings which can be produced by the recursive method of Fig. 4. Beginning with generation 1 we always have a choice of two square tiles to use at the center of our evolving tiling: one with clockwise orientation, the other counterclockwise. The choice will determine how the rhombic tiles are positioned. Therefore, such a tiling is the unique outcome of an infinite sequence of choices among two possibilities c and cc, *i.e.* is in one-to-one correspondence with an infinite sequence of two symbols. As is well known, the collection of all such sequences is uncountably infinite.

A simple modification of the pattern of Fig. 1(a) gives quasiperiodic tilings which have only 2-fold rotational symmetry (Fig. 6).



(a)

(b)

Fig. 6. (a) using second-generation tiles, a modification of Fig. 1(a) results in (b) a third-generation square with 2-fold rotational symmetry.

In concluding this section we mention the case of 8-fold rotational symmetry (Watanabe, Ito & Soma, 1987) obtained by using eight nth-generation rhombi to form an 8-pointed star, Fig. 7.

2. Three- and sixfold rotational symmetry

The general outline of our method has been given in §1. We create self-similar tiles by solving an eigenvalue problem that tells us how many of each type of tile to use and sheds light on how to configure the boundaries of the next generation's tiles. In this section we construct quasiperiodic tilings with 3-fold and 6-fold rotational symmetry using three unit-edged prototiles (30, 60 and 90° rhombi).

The eigenvector of prototile areas is $\mathbf{a} = (1/2, 3^{1/2}/2, 1)^T$. This time we specify the maximum eigenvalue $\lambda = 2 + 2\cos(\pi/6)$ in advance to take advantage of the aforementioned property of palindromy (also exploited heavily by Watanabe *et al.* 1987). That is, we are looking for first-generation tiles that have a half-60° rhombus sandwiched between two unit prototiles along the edges, Fig. 8. The length of the cut diagonal is $2\cos(\pi/6)$, hence each first-generation tile will have edge length $2 + 2\cos(\pi/6) = \lambda$. Now, however, it is not easy to find the integer-valued matrix M needed to construct the first-generation tiles. In fact, a mathematical theory is necessary to determine M. An introduction to this theory is presented by Clark & Suryanarayan (1991). However, the reader can easily read the rows of M^2 from Figs. 8(a), (b) and (c).

For example, Fig. 8(a) indicates that the first row of M^2 is (5, 4, 1), for the five black, four gray and one white prototiles, respectively. At first glance the

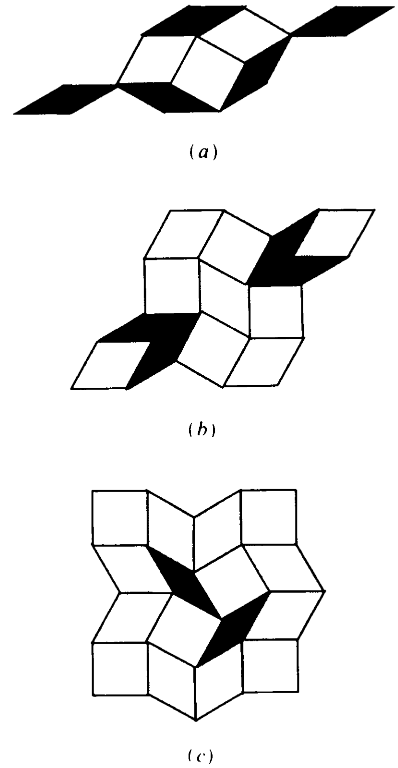


Fig. 8. First-generation tiles.

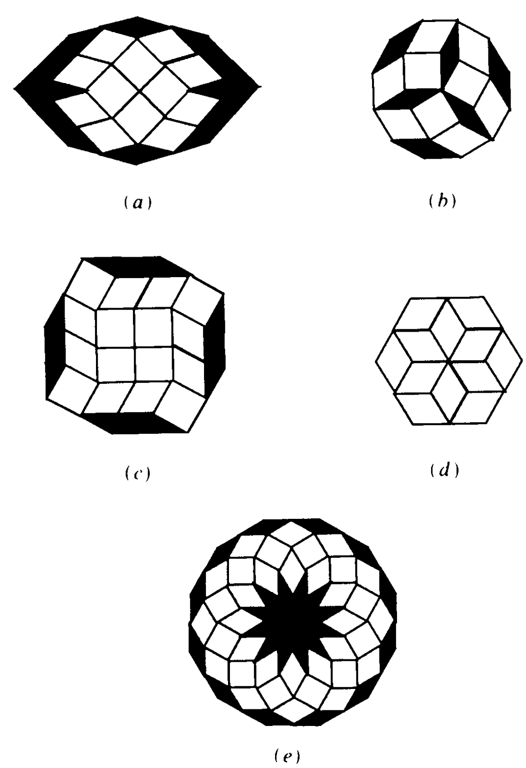


Fig. 9. First-generation models with 2-, 3-, 4-, 6- and 12-fold rotational symmetry.

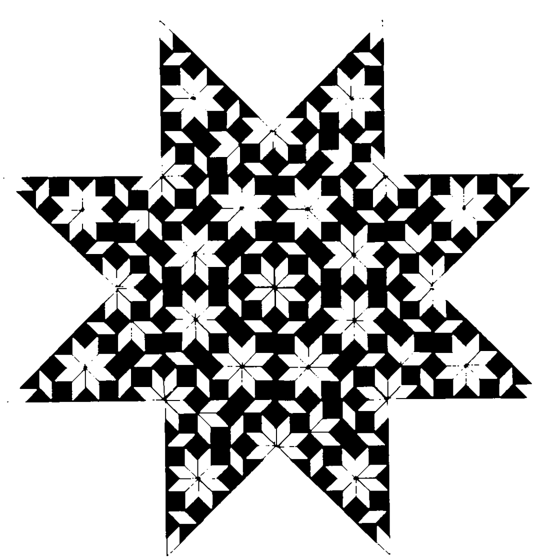


Fig. 7. 8-fold rotational symmetry.

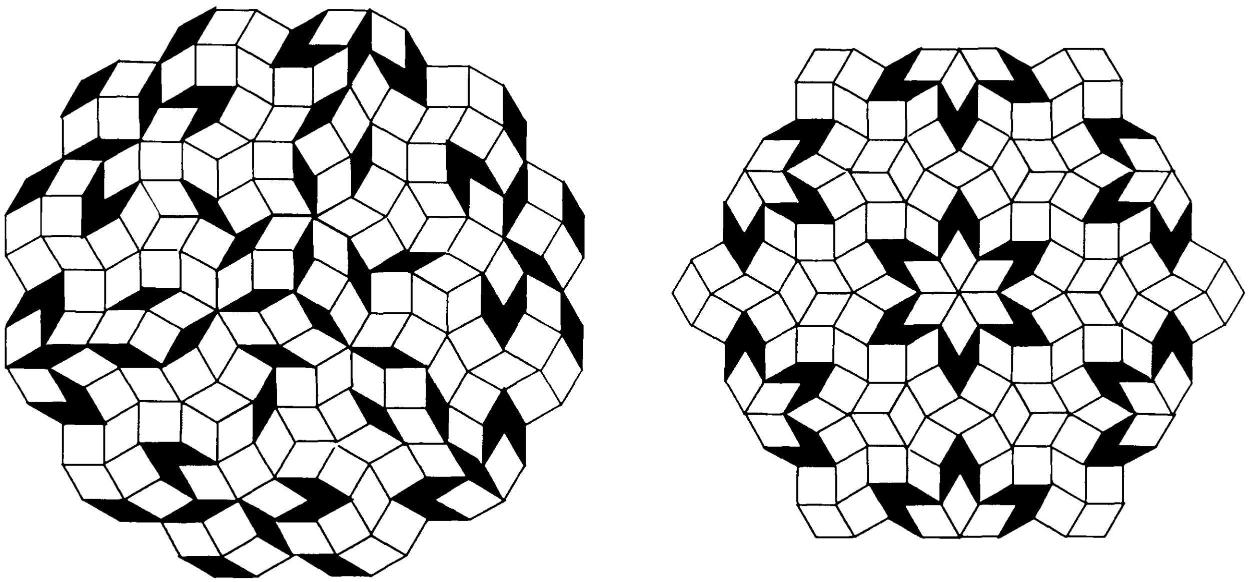


Fig. 10. (*a*) 3-fold rotational symmetry, 2nd generation. (*b*) 6-fold rotational symmetry, 2nd generation.

projections and indentations of Fig. 8(*c*) make it look like a possible violation, but in fact it is simply a generalized square, *i.e.* a new aspect of the solution. The reader will see how the projections and indentations work in Fig. 10.

As first-generation models for tilings with 3- and 6-fold rotational symmetry we use the configurations of Figs. 9(*b*) and (*d*), respectively. We go to the next generation, Fig. 10, by replacing the prototiles of Fig. 9 with the tiles of Fig. 8. We could do the same in Figs. 9(*a*), (*c*), (*e*).

The proofs of the validity (*i.e.* absence of interior edge violations) of the tilings of the infinite plane obtained by substituting higher-generation tiles into the configurations of Fig. 9 and passing to the limit, as well as the uncountable infinity of these tilings, follow the line of arguments given in §1.

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Polarization Effects on X-ray Multiple Diffraction

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Abstract

The effects of polarization on X-ray multiple diffraction are investigated experimentally. Polarized and unpolarized incident beams are used in multiple diffraction experiments for GaAs and Ge crystals. All the three-beam diffractions in a 360° azimuthal rotation of the crystals are analyzed for diffracted intensities and phase determination, based on the article by Chang & Tang [*Acta Cryst.* (1988), **A44**,

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